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A note on the existence of graded extensions of Poisson brackets

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In this note we discuss the existence and projectability of graded extensions of ordinary Poisson brackets. We will show that there are topological obstructions to both problems. To prove it we use a new algebraic characterization of graded Poisson brackets on graded manifolds based on a characterization of derivations on the exterior algebra of a vector bundle.

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1. The extension problem

A graded Poisson manifold (or a Poisson supermanifold) [1] is a pair $(P, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of Poisson superalgebras over P and $(P, \mathcal{A})$ is a supermanifold in the sense of Kostant [5]. The graded Lie algebra structure $\{\ ,\}$ of the sheaf $\mathcal{A}$ will be called the graded Poisson bracket of the supermanifold. Most of the results in this note can be extended easily to odd Poisson brackets but we will restrict in what follows to even Poisson brackets. We should remark that a graded Poisson bracket $\{\ ,\}$ induces a Poisson bracket on the underlying manifold P such that the augmentation map restricted to $\mathcal{A}_0$ is a Poisson morphism.

As is well known, any smooth supermanifold $(M, \mathcal{A})$ is simple or split, i.e. isomorphic to a supermanifold whose sheaf of superfunctions is the sheaf of sections of the exterior algebra $\wedge E$ of a vector bundle $E \to M$. The vector bundle E is called the conormal bundle of the supermanifold $(M, \mathcal{A})$. Given a Poisson manifold P with a Poisson structure $\{\ ,\}_P$ and a vector bundle $E \to P$, we can consider the existence of a graded Poisson bracket $\{\ ,\}$ on the sheaf $\Gamma(\wedge E)$ such that $\{F, G\} = \{F, G\}_P$ for all $F, G \in C^\infty(P)$. We can state this problem in terms of abstract graded manifolds as follows.

Extension problem. Let $(P, \mathcal{A})$ be a graded manifold over the Poisson manifold $(P, \{\,,\}_P)$. Does there exist a graded Poisson bracket $\{\,,\}$ on $(P, \mathcal{A})$ and a splitting $p : C^\infty \to \mathcal{A}$ such that $\{p(F), p(G)\} = p(\{F, G\}_P)$ for all $F, G \in C^\infty$?

Notice that, if the extension problem has a solution, then the splitting p is a Poisson retraction. A graded Poisson bracket on $(P, \mathcal{A})$ will be called projectable if it is a solution of the extension problem for the induced Poisson bracket on P, and it will also be said that a projectable Poisson bracket is an extension along a retraction map of an ordinary Poisson bracket. It happens that the existence of projectable Poisson graded brackets on $(P, \mathcal{A})$ depends on the topological properties of the conormal bundle E, as the following theorem shows.

Theorem 1. *Let $(P, \mathcal{A})$ be a graded manifold with conormal bundle E and let $\{\,,\}_P$ be a constant rank Poisson bracket on P; then there exists a projectable graded Poisson bracket on $\mathcal{A}$ inducing the Poisson bracket $\{\,,\}_P$ on P if and only if there exists a connection on E whose curvature vanishes along the symplectic leaves of $\{\,,\}_P$.*

Corollary 2. *Any projectable symplectic supermanifold $(M, \mathcal{A})$ is isomorphic to a flat symplectic supermanifold $(M, \Gamma(\wedge E))$.*

It was proved in ref. [4] and refined later in ref. [8], that any supermanifold whose underlying differentiable manifold has an exact symplectic structure can be endowed with a graded symplectic structure. It is an immediate consequence of corollary 2 that these symplectic supermanifolds cannot be projectable unless their conormal bundle is flat. In the same way of reasoning it is noticeable that the canonical graded symplectic structure on the cotangent supermanifold $(T^*M, T^*\mathcal{A})$ of a graded manifold $(M, \mathcal{A})$ (see ref. [5]) is projectable iff the conormal bundle E of $\mathcal{A}$ is flat (notice that the conormal bundle of $T^*\mathcal{A}$ is $E \oplus E^*$). However, the graded version of the Darboux theorem assures that we can find canonical supercoordinates for graded symplectic structures. If the graded symplectic form is not projectable these canonical supercoordinates will never define a splitting of the underlying graded manifold. It is also remarkable that there are no graded extensions of Poisson brackets to the Cartan algebra of a given Poisson manifold unless the tangent bundle of the manifold is flat (see for example the discussions in refs. [2] and [6]).

2. A characterization of graded Poisson brackets

A derivation D of $\mathcal{A}$ is called a graded vector field on $(P, \mathcal{A})$ and the sheaf of derivations is naturally a sheaf of graded Lie algebras. Given a splitting of $\mathcal{A}$ we can identify it with the sheaf $\Gamma(\,, \wedge E)$. The Poisson superbracket $\{\,,\}$ is a derivation on each argument and consequently, for any superfunction h on P, the linear operator $\{h, \cdot\}$ defines a supervector field D_h called the Hamiltonian supervector field associated to the Hamiltonian function h.

Let $TP \to P$ be the tangent bundle over P and $\Gamma(\wedge E \otimes TP)$ the $\Gamma(\wedge E)$-module of smooth sections of $\wedge E \otimes TP$. Let ∇ be a linear connection in E. If $K = f_{(k)} \otimes X \in \wedge^k(E) \otimes TP$, we define the endomorphism $\nabla_K : \Gamma(\wedge E) \to \Gamma(\wedge E)$ by $\nabla_K u = f_{(k)} \nabla_X u$, where $u \in \Gamma(\wedge E)$, and if $K \in \Gamma(\wedge E \otimes TP)$, we define ∇_K by its linear extension. It is obvious that ∇_K thus defined is a derivation and we call it the proper derivation associated to K through ∇.

Now, we shall define another type of derivations, the algebraic ones. Let $E^* \to P$ be the dual bundle of E, and let $\Gamma(\wedge E \otimes E^*)$ be the $\Gamma(\wedge E)$-module of smooth sections of $\wedge E \otimes E^*$. If $L = f_{(k)} \otimes \alpha \in \wedge^k(E) \otimes \Gamma(E^*)$, we define the endomorphism $i_L : \Gamma(\wedge E) \to \Gamma(\wedge E)$ by $i_L u = f_{(k)} i_\alpha u$, where $u \in \Gamma(\wedge E)$ and i_α is the interior multiplication. If $L \in \Gamma(\wedge E \otimes E^*)$, we define i_L by its linear extension. It is clear that i_L is a derivation and acts trivially on the smooth functions on P. We will call i_L the algebraic derivation associated to L.

With the help of a connection ∇ on E, we can characterize the graded vector fields on $(P, \Gamma(\ , \wedge E))$. Let D be a derivation on $\Gamma(\wedge E)$; then there are unique fields $K \in \Gamma(\wedge E \otimes TP)$ and $L \in \Gamma(\wedge E \otimes E^*)$ such that $D = \nabla_K + i_L$ [7,3].

The graded commutator of two algebraic derivations, $[i(L_1), i(L_2)]$, is again an algebraic derivation; thus, there is a unique element $[L_1, L_2]_{\mathrm{RN}} \in \Gamma(\wedge E \otimes E^*)$ such that $[i(L_1), i(L_2)] = i([L_1, L_2]_{\mathrm{RN}})$. The field $[L_1, L_2]_{\mathrm{RN}} \in \Gamma(\wedge E \otimes E^*)$ is called the Richardson–Nijenhuis bracket of L_1 and L_2. With this bracket the space $\Gamma(\wedge^{*+1} E \otimes E^*)$ becomes a graded Lie algebra. When $E = T^*P$ this bracket is the usual Richardson–Nijenhuis bracket.

The graded commutator of two proper derivations, $[\nabla_{K_1}, \nabla_{K_2}]$, is again a derivation; then there are unique elements $[K_1, K_2]_{\mathrm{FN}} \in \Gamma(\wedge E \otimes TP)$ and $R(K_1, K_2) \in \Gamma(\wedge E \otimes E^*)$ such that $[\nabla_{K_1}, \nabla_{K_2}] = \nabla_{[K_1, K_2]_{\mathrm{FN}}} + i_{R(K_1, K_2)}$. We shall call $[K_1, K_2]_{\mathrm{FN}} \in \Gamma(\wedge E \otimes TP)$ the Frölicher–Nijenhuis bracket of K_1 and K_2 with respect to ∇, but notice that it is not a Lie bracket and that it does not agree with the usual Frölicher–Nijenhuis bracket when $E = T^*P$. It is also noticeable that for vector fields X, Y we have that $[X, Y]_{\mathrm{FN}} = [X, Y]$, the usual Lie bracket, and $R(X, Y)$ is the curvature tensor of the linear connection ∇ acting on X, Y.

The graded Hamiltonian vector field D_f is a derivation of degree $|f|$ and again, given a fixed connection ∇ on E, there exist unique $K_f \in \Gamma(\wedge^{|f|} E \otimes TP)$ and $L_f \in \Gamma(\wedge^{|f|+1} E \otimes E^*)$ such that $D_f = \nabla_{K_f} + i_{L_f}$.

Let F be a vector bundle over P; then $\Gamma(\wedge E \otimes F)$ can be seen as a $\Gamma(\wedge E)$-module. A map $D : \Gamma(\wedge E) \to \Gamma(\wedge E \otimes F)$ is called a derivation of degree $|D|$ if $D(fg) = D(f)g + (-1)^{|D||f|} f D(g)$. Then it is simple to show that the map $K : \Gamma(\wedge E) \to \Gamma(\wedge E \otimes TP)$ defined by $f \mapsto K_f$ is a derivation of degree 0, and the map $L : \Gamma(\wedge E) \to \Gamma(\wedge E \otimes E^*)$ defined by $f \mapsto (-1)^{|f|} L_f$ is a derivation of degree 1. Thus, given a pair of derivations as before, (K, L), we can construct a bracket on $\Gamma(\wedge E)$ satisfying Leibniz' rule by means of the formula

$$\{f, g\} = \nabla_{K_f} g + (-)^{|f|} i_{L_f} g. \tag{1}$$

The characterization of derivations is extended by the following

Proposition 3. *Let D be a derivation from $\Gamma(\wedge E)$ into $\Gamma(\wedge E \otimes F)$ and let ∇ be a linear connection in E. Then there are unique fields $\phi \in \Gamma(\wedge E \otimes F \otimes TP)$ and $\psi \in \Gamma(\wedge E \otimes F \otimes E^*)$ such that $D = \nabla_\phi + i_\psi$, where ∇_ϕ and i_ψ are derivations from $\Gamma(\wedge E)$ into $\Gamma(\wedge E \otimes F)$ defined analogously to derivations of $\Gamma(\wedge E)$.*

Let us apply this characterization to the two derivations associated to a graded Poisson bracket. K is a derivation from $\Gamma(\wedge E)$ into $\Gamma(\wedge E \otimes TP)$; then, by proposition 3, it defines uniquely two tensors $\Pi \in \Gamma(\wedge E \otimes TP \otimes TP)$ and $\Sigma \in \Gamma(\wedge E \otimes TP \otimes E^*)$. On the other hand L is a derivation from $\Gamma(\wedge E)$ into $\Gamma(\wedge E \otimes E^*)$; $F = E^*$ in the notation of the previous proposition, thus it is defined by two more tensors, $\Phi \in \Gamma(\wedge E \otimes E^* \otimes TP)$ and $S \in \Gamma(\wedge E \otimes E^* \otimes E^*)$. Now, it is easy to check that the graded commutativity of the graded Poisson bracket implies that $\Pi \in \Gamma(\wedge E \otimes \wedge^2 TP)$, $S \in \Gamma(\wedge E \otimes S^2 E^*)$ and that Φ is the transposition of Σ. We get then the following

Proposition 4. *Let $\{\ ,\ \}$ be a graded bracket on $\Gamma(\wedge E)$ satisfying graded commutativity and Leibniz' rule; then, given a connection in E, the graded bracket is uniquely determined by three tensors $\Pi \in \Gamma(\wedge E \otimes \wedge^2 TP)$, $\Sigma \in \Gamma(\wedge E \otimes TP \otimes E^*)$ and $S \in \Gamma(\wedge E \otimes S^2 E^*)$.*

The bracket constructed using eq. (1) satisfies Leibniz' rule but not necessarily the graded Jacobi identity. Writting the graded Jacobi identity in terms of the derivations ∇_K, i_L, we get

$$K_{\{f,g\}} = [K_f, K_g]_{\mathrm{FN}} + i_{L_f} K_g - (-1)^{|f||g|} i_{L_g} K_f, \tag{2}$$

$$L_{\{f,g\}} = R(K_f, K_g) + \nabla_{K_f} L_g - (-1)^{|f||g|} \nabla_{K_g} L_f + [L_f, L_g]_{\mathrm{RN}}, \tag{3}$$

and R denotes the linear extension to $\Gamma(\wedge E \otimes TP)$ of the curvature of the connection ∇. Let us write K_f^n (resp. L_f^n) for the part of degree n of K_f (resp. L_f). The part of degree zero of (2) is

$$K^0_{\{f,g\}} = [K_f^0, K_g^0] + i_{L_f^1} K_g^0 - i_{L_g^1} K_f^0, \tag{4}$$

and the part of degree 1 of (3) is

$$L^1_{\{f,g\}} = R(K_f^0, K_g^0) + \nabla_{K_f^0} L_g^1 - \nabla_{K_g^0} L_f^1 + [L_f^1, L_g^1]_{\mathrm{RN}}. \tag{5}$$

Proof of the extension problem. Let us suppose that there exists a connection ∇ on E such that $L^1_f = 0$ for all $f \in \Gamma(\wedge E)$. For such a connection we get immediately from (5) that $0 = R(K^0_f, K^0_g)$. Moreover, since the graded Poisson bracket is an extension, then K^0_f is the Hamiltonian vector field X_{f_0} associated to $f_0 \in C^\infty(P)$ using the Poisson bracket $\{\ ,\}_P$. Therefore the curvature of the connection vanishes on the vector fields of the characteristic distribution of π_0, i.e., on the symplectic leaves of P.

Let us prove now that there exists a connection such that $L^1_f = 0$ for all $f \in \Gamma(\wedge E)$. Let C be the characteristic distribution defined by the constant rank Poisson structure $\{\ ,\ \}_P$ and let D be a complementary distribution such that $C \oplus D = TP$. Let K, L be the operators defined by the graded Poisson bracket extension. Let us define a one-form on C with values in $E \oplus E^*$ by means of $A(X_F) = L^1_F$ for all $F \in C^\infty(P)$. Then A is defined on all vector fields in C because the Hamiltonian vector fields X_F generate the subbundle C of TP. Then, finally let us define $A(X) = A(X^C)$ for all vector fields X in P, where $X = X^C + X^D$, $X^C \in C$ and $X^D \in D$. Because of the tensoriality of L we have a well-defined $A : TP \to E \otimes E^*$ that depends only on the choice of D. Let us define another connection ∇', by writing $\nabla'_X = \nabla_X + i_{A(X)}$; then it is clear that we have $(L'_f)^1 = 0$ for all $f \in \Gamma(\wedge E)$.

Conversely, let us suppose that there exists a connection ∇ on E whose curvature vanishes along symplectic leaves of the Poisson manifold P. In order to define a graded Poisson bracket we must specify two derivations K and L satisfying eqs. (2), (3). Let $K : \Gamma(\wedge E) \to \Gamma(\wedge E \otimes TP)$ be the derivation ∇_π, where π is the Poisson tensor field associated to the Poisson bracket $\{\ ,\ \}_P$, and let $L = 0$. In a coordinate basis, if

$$\{F, G\}_P = \pi^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}$$

for $F, G \in C^\infty(P)$, the action of the derivation K on $f \in \Gamma(\wedge E)$ is given by

$$K_f = \nabla_\pi f = \pi^{ij} \nabla_i f \otimes \partial / \partial x^j,$$

where ∇_i denotes $\nabla_{\partial/\partial x^i}$. Therefore, the graded Poisson bracket of $f, g \in \Gamma(\wedge E)$ is given using (1) by

$$\{f, g\} = D_f g = \nabla_{K_f} g = \pi^{ij} \nabla_i f \nabla_j g.$$

Graded commutativity is thus immediate, as well as Leibniz' rule due to the choice of derivations. The only point that remains to be proved is the graded Jacobi identity. We have to check formulas (2) and (3). With our choice of operators K, L and having in mind that the curvature vanishes on sympletic leaves, the second one reduces to a trivial identity. Indeed,

$$R(K_f, K_g) = \nabla_i f \nabla_j g R(K_{x^i}, K_{x^j}) = 0,$$

and a simple computation shows that $K_{\{f,g\}} = [K_f, K_g]_{\mathrm{FN}}$, for all $f, g \in \Gamma(\wedge E)$.

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